Journal of Approximation Theory **96**, 378–398 (1999)
Article ID jath.1997.3248, available online at http://www.idealibrary.com on IDEAL®

Estimation of Norms of Multivariate Polynomials with Integral Coefficients*

Francisco Luquin

Departamento de Matemáticas, Facultad de Ciencias, Universidad del País Vasco, Apdo. 644, 48080 Bilbao, Spain
E-mail: mtplumaf@lg.ehu.es

and

Concepción Besga

Departamento de Matemática Aplicada, E.U.I.T.I. e. I.T.T., Universidad del País Vasco, Nieves Cano 12, 01006 Vitoria, Spain

Communicated by Peter B. Borwein

Received September 10, 1997; accepted in revised form January 9, 1998

DEDICATED TO THE MEMORY OF PROFESSOR EMILIANO APARICIO

Using Fekete's method we obtain estimates for the L_p-norms of minimal integral generalized multivariate polynomials. We particularize these estimates for the cases of ordinary polynomials and quasi-polynomials. We also show the existence of a limit in the minimal quadratic deviations from zero for univariate integral polynomials. © 1999 Academic Press

1. INTRODUCTION AND MAIN RESULTS

Let I be an interval of the real line, let H be a vector space of real functions defined on I containing the polynomials and let $\|\cdot\|$ be a norm defined on H. Any polynomial $Q(x) \not\equiv 0$ with integral coefficients such that

$$\|Q\| < 1$$

is called a fundamental polynomial of I with respect to $\|\cdot\|$ (see [5]).

The existence of fundamental polynomials plays an important role in the Theory of approximation of functions by polynomials with integral coefficients. The first result in this direction was obtained by Hilbert [14] who

* Supported by the University of the Basque Country.

0021-9045/99 $30.00

established the existence of fundamental polynomials with respect to the L_2-norm whenever the length of the interval is less than 4. The analogous result for the uniform norm follows from a well-known theorem of Fekete [9]. Fekete's method was extended by Aparicio (see [1, 3, 4]) to the estimation of uniform and L_2-norms of minimal integral generalized polynomials in several variables on parallelepipeds in Euclidean spaces. Another result in uniform norm for ordinary multivariate polynomials was obtained by Zirnova (see [20]). The extension of Fekete's theorem to L_p-norms and univariate generalized polynomials is given in [15].

In this paper, we obtain estimates for the L_p-norms of minimal integral generalized multivariate polynomials. Before stating our main results we shall introduce the notation which will be used throughout the paper.

Let $p_k(x_k)$ and $u_k(x_k)$, $k=1, ..., m$, be weight functions defined on the segment $[a_k, b_k]$ (non-negative summable functions which assume the value zero only on a set of measure zero) and denote by $L^p{}_{v(x)}([a, b])$, $p \geqslant 1$, the class of all real functions $f(x)$ defined on $[a, b]$ for which the product $v(x)\,|f(x)|^p$ is summable on the segment $[a, b]$. For each k, let $\phi_{k;1}(x_k), ..., \phi_{k;n_k}(x_k), ...,$ be a finite or denumerable infinite system of linearly independent functions belonging to $L^2{}_{p_k(x_k)}([a_k, b_k]) \cap L^p{}_{u_k(x_k)}([a_k, b_k])$. Applying the Schmidt orthogonalization procedure, we obtain an orthonormal system $\omega_{k;1}(x_k), ..., \omega_{k;n_k}(x_k), ...,$ with respect to the weight function $p_k(x_k)$ on $[a_k, b_k]$, which satisfies the relations (see [17])

$$\phi_{k;i}(x_k) = \sum_{j=1}^{n_k} b_{k;ij}\omega_{k;j}(x_k), \qquad i=1, ..., n_k,\ b_{k;ij}=0 \text{ if } i<j, \tag{1.1}$$

where the matrix of the coefficients $A_{k;n_k}$ is the lower triangular matrix given by

$$A_k := A_{k;n_k} = (b_{k;ij}), \qquad \text{with} \quad b_{k;ii} = (\Delta_{k;i}/\Delta_{k;i-1})^{1/2}, \tag{1.2}$$

$\Delta_{k;i}$ being the Gram determinant of the system of functions $\{\phi_{k;r}(x_k)\}_{r=1}^{i}$, $\Delta_{k;0} := 1$. From (1.1) we have

$$b_{k;ii} = \int_{a_k}^{b_k} p_k(x_k)\,\phi_{k;i}(x_k)\,\omega_{k;i}(x_k)\,dx_k$$

and from (1.2)

$$|A_k| = b_{k;11} \cdots b_{k;n_kn_k} = \Delta_{k;n_k}^{1/2} =: \Delta_{n_k}^{1/2}.$$

Our first result generalizes Theorem 1 in [15] to several variables:

THEOREM 1.1. *There exists a generalized polynomial*

$$Q_{n_1 \cdots n_m}(x_1, ..., x_m) = \sum_{k_1=1}^{n_1} \cdots \sum_{k_m=1}^{n_m} \alpha_{k_1 \cdots k_m} \phi_{1; k_1}(x_1) \cdots \phi_{m; k_m}(x_m), \tag{1.3}$$

with rational integral coefficients, not simultaneously zero, such that

$$I_{n_1 \cdots n_m} := \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} u_1(x_1) \cdots u_m(x_m) \, |Q_{n_1 \cdots n_m}(x_1, ..., x_m)|^p \, dx_1 \cdots dx_m$$
$$\leqslant \prod_{i=1}^{m} \left(n_i^{p-1} \Delta_{n_i}^{p/(2n_i)} \sum_{l_i=1}^{n_i} N_{i; l_i} \right), \tag{1.4}$$

where

$$N_{i; j} := \int_{a_i}^{b_i} u_i(x_i) \, |\omega_{i; j}(x_i)|^p \, dx_i, \qquad i = 1, ..., m. \tag{1.5}$$

The proof of Theorem 1.1 is given in Section 2. In Section 3, we show that, in some particular cases, the estimate obtained in Theorem 1.1 is asymptotically optimal. In Section 4, Theorem 1.1 is applied to give upper bounds for L_p-norms of ordinary polynomials.

Our second result extends to L_p-norms a result of Aparicio (see [1, 4]) for the L_2-norm. It is stated as follows:

THEOREM 1.2. *There exists a generalized polynomial*

$$Q_h(x_1, ..., x_m) = \sum_{\substack{k_1 + \cdots + k_m \leqslant h \\ k_i \geqslant 1}} \alpha_{k_1 \cdots k_m} \phi_{1; k_1}(x_1) \cdots \phi_{m; k_m}(x_m), \qquad m \geqslant 2, \tag{1.6}$$

with rational integral coefficients, not simultaneously zero, such that

$$I_h := \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} u_1(x_1) \cdots u_m(x_m) \, |Q_h(x_1, ..., x_m)|^p \, dx_1 \cdots dx_m$$
$$\leqslant \left[\prod_{r=1}^{h-m+1} \left(\prod_{k=1}^{m} \Delta_{k; r}^{1/2} \right)^{\binom{h-r-1}{m-2}} \right]^{p \binom{h}{m}^{-1}}$$
$$\times \binom{h}{m}^{p-1} \sum_{\substack{l_1 + \cdots + l_m \leqslant h \\ l_i \geqslant 1}} (N_{1; l_1} \cdots N_{m; l_m}), \tag{1.7}$$

where $N_{i; j}$ is given by (1.5).

Theorem 1.2 is shown in Section 5. Section 6 contains applications of Theorem 1.2 to ordinary polynomials.

On the other hand we consider quasi-polynomials with rational integral coefficients of the form

$$Q_{n_1 \cdots n_m}(x_1, ..., x_m) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_m=0}^{n_m} \alpha_{k_1 \cdots k_m} x_1^{k_1^{\gamma_1}} \cdots x_m^{k_m^{\gamma_m}}, \tag{1.8}$$

where $\gamma_1, ..., \gamma_m$ are certain positive real numbers. We set

$$v_{n_1 \cdots n_m}^{-2(n_1 + \cdots + n_m)}(\gamma_1, ..., \gamma_m) := \inf_{Q_{n_1 \cdots n_m}} \int_0^1 \cdots \int_0^1 Q^2_{n_1 \cdots n_m}(x_1, ..., x_m) \, dx_1 \cdots dx_m ,$$

where the infimum is extended over the class of all non-trivial quasi-polynomials of the type (1.8) and we write

$$v(\gamma_1, ..., \gamma_m) := \lim_{\substack{n_1 \to \infty \\ \cdots \\ n_m \to \infty}} v_{n_1 \cdots n_m}(\gamma_1, ..., \gamma_m).$$

The following result generalizes to several variables a result of Aparicio [2]:

THEOREM 1.3. *Assume that* $n_k = n$, *for all* k. *Then, the following inequality holds*:

$$\mathbf{v}(\gamma_1, ..., \gamma_m) := \lim_{n \to \infty} v_{n \cdots n}(\gamma_1, ..., \gamma_m) \geqslant e^{1/m \sum_{k=1}^m J(\gamma_k)}$$

where

$$J(\gamma_k) := \sum_{r=0}^{\infty} \frac{1}{(2r+1)[(2r+1)\gamma_k + 1]}.$$

The proof of Theorem 1.3 is given in Section 7. In Section 8 we obtain bounds for $\mathbf{v}(\gamma_1, ..., \gamma_m)$ in some particular cases.

Finally, Section 9 is concerning with univariate integral polynomials. Let $p(x)$ be a weight function defined on the interval $[a, b]$ and let $\{\omega_k(x)\}$ be the corresponding orthonormal system of polynomials. We write

$$K_n[a, b] := \max_{\substack{a \leqslant t \leqslant b \\ a \leqslant x \leqslant b}} |K_n(t, x)|,$$

where

$$K_n(t, x) := \sum_{k=0}^{n} \omega_k(t) \, \omega_k(x).$$

We consider the set

$$\mathscr{P}[a, b] := \{ p(x) \text{ weight function} : \lim_{n \to \infty} K_n^{1/n}[a, b] = 1 \}.$$

We define

$$\rho[a, b] := \lim_{n \to \infty} \rho_n[a, b], \qquad \rho_n^{-n}[a, b] := \min_{P \in H_n^e} \max_{a \leqslant x \leqslant b} |P(x)| \quad (1.9)$$

and

$$\tau_n^{-2n}[a, b] := \min_{Q \in H_n^e} \int_a^b Q^2(x)\, p(x)\, dx,$$

where H_n^e is the class of rational integral polynomials of degree less than or equal to n, not identically zero. We prove the following theorem (for a similar result, see [6]):

THEOREM 1.4. *Let $p(x) \in \mathscr{P}[a, b]$. Then, the limit*

$$\tau[a, b] := \lim_{n \to \infty} \tau_n[a, b] \tag{1.10}$$

exists and we have

$$\tau[a, b] = \rho[a, b], \tag{1.11}$$

where $\rho[a, b]$ is defined by (1.9).

2. PROOF OF THEOREM 1.1

By substituting in the left-hand side of inequality (1.4) the corresponding expressions (1.1) for the functions $\phi_{1;\, k_1}(x_1), ..., \phi_{m;\, k_m}(x_m)$ we obtain

$$I_{n_1 \cdots n_m} = \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} u_1(x_1) \cdots u_m(x_m)$$
$$\times \left| \sum_{k_1=1}^{n_1} \cdots \sum_{k_m=1}^{n_m} \alpha_{k_1 \cdots k_m} \sum_{l_1=1}^{n_1} b_{1;\, k_1 l_1} \omega_{1;\, l_1}(x_1) \cdots \right.$$
$$\left. \times \sum_{l_m=1}^{n_m} b_{m;\, k_m l_m} \omega_{m;\, l_m}(x_m) \right|^p dx_1 \cdots dx_m.$$

By changing the order of summation and denoting by

$$L_{l_1 \cdots l_m} := \sum_{k_1=1}^{n_1} \cdots \sum_{k_m=1}^{n_m} \alpha_{k_1 \cdots k_m} b_{1;\, k_1 l_1} \cdots b_{m;\, k_m l_m}$$
$$(1 \leqslant l_i \leqslant n_i,\ i = 1, ..., m) \tag{2.1}$$

we obtain

$$I_{n_1 \cdots n_m} \leqslant \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} u_1(x_1) \cdots u_m(x_m) \times \left[\sum_{l_1=1}^{n_1} \cdots \sum_{l_m=1}^{n_m} |L_{l_1 \cdots l_m}|\, |\omega_{1;\, l_1}(x_1)| \cdots |\omega_{m;\, l_m}(x_m)| \right]^p dx_1 \cdots dx_m. \tag{2.2}$$

We consider the system (2.1) of $n_1 \cdots n_m$ linear forms with $\{\alpha_{k_1 \cdots k_m}\}$ as unknowns. We claim that the determinant Δ of the matrix of the linear system (2.1) is

$$\Delta = \Delta_{n_1}^{n_2 \cdots n_m/2} \cdots \Delta_{n_m}^{n_1 \cdots n_{m-1}/2}. \tag{2.3}$$

This follows by observing that such a matrix is the matrix corresponding to a change from the $\phi_{1;\, k_1}(x_1) \otimes \cdots \otimes \phi_{m;\, k_m}(x_m)$, $1 \leqslant k_1 \leqslant n_1, ..., 1 \leqslant k_m \leqslant n_m$, to the $\omega_{1;\, l_1}(x_1) \otimes \cdots \otimes \omega_{m;\, l_m}(x_m)$, $1 \leqslant l_1 \leqslant n_1, ..., 1 \leqslant l_m \leqslant n_m$, basis in the tensor product $U_1 \otimes \cdots \otimes U_m$, where U_i is the linear subspace spanned by $\phi_{i;\, 1}(x_i), ..., \phi_{i;\, n_i}(x_i)$.

It is known [12] that the matrix $A_1 \otimes \cdots \otimes A_m$ describing the change of these bases is the Kronecker product of the matrices $A_1, ..., A_m$, and the determinant of this matrix of the transformation between them is

$$\Delta = |A_1 \otimes \cdots \otimes A_m| = |A_1|^{n_2 \cdots n_m} \cdots |A_m|^{n_1 \cdots n_{m-1}}.$$

Therefore, (2.3) follows from the fact that $|A_k| = \Delta_{n_k}^{1/2}$.

According to the Minkowski's linear forms Theorem (see [19]), there exists a system of rational integral coefficients $\{\alpha_{k_1 \cdots k_m}\}$, not simultaneously zero such that

$$|L_{l_1 \cdots l_m}| \leqslant \Delta^{1/(n_1 \cdots n_m)} = \Delta_{n_1}^{1/(2n_1)} \cdots \Delta_{n_m}^{1/(2n_m)}. \tag{2.4}$$

Substituting (2.4) in (2.2) and taking into account that

$$(a_1 + \cdots + a_n)^p \leqslant n^{p-1}(a_1^p + \cdots + a_n^p), \tag{2.5}$$

the inequality (1.4) follows. ∎

Remark 2.1. In the case of the quadratic norm and $p_k(x_k) = u_k(x_k)$ for all k, each system $\{\omega_{k;j}(x_k)\}_{j=1}^{n_k}$ is orthonormal with respect to each weight function $u_k(x_k)$. Therefore, we get

$$I_{n_1 \cdots n_m} \leqslant \prod_{i=1}^{m} (n_i \Delta_{n_i}^{1/n_i}).$$

Remark 2.2. If the functions $u_k(x_k)$ and $\{\phi_{k;i}(x_k)\}$ belong to $C([a_k, b_k])$, then

$$\max_{\substack{a_1 \leqslant x_1 \leqslant b_1 \\ \cdots \\ a_m \leqslant x_m \leqslant b_m}} u_1(x_1) \cdots u_m(x_m) \, |Q_{n_1 \cdots n_m}(x_1, ..., x_m)| \leqslant \prod_{i=1}^{m} \left(\Delta_{n_i}^{1/(2n_i)} \sum_{l_i=1}^{n_i} \|\omega_{i;\, l_i}\| \right),$$

where $\|\omega_{i;j}\| = \max_{a_i \leqslant x_i \leqslant b_i} u_i(x_i) \, |\omega_{i;j}(x_i)|$. For similar results, see [4, 18].

Remark 2.3. In [20], trigonometric polynomials are used to show a similar result to Theorem 1.1 for $p = \infty$ and $u_k(x_k) = 1$, for all k.

On the other hand, given m non-negative integer numbers $n_1, ..., n_m$, we may consider the value $\sigma_{n_1 \cdots n_m}$ defined by

$$\sigma_{n_1 \cdots n_m}^{-p(n_1 + \cdots + n_m)} := \inf_{Q_{n_1 \cdots n_m}} \int_a^b \cdots \int_a^b u_1(x_1) \cdots u_m(x_m) \times |Q_{n_1 \cdots n_m}(x_1, ..., x_m)|^p \, dx_1 \cdots dx_m, \tag{2.6}$$

where the infimum is extended over the class of all non-trivial generalized polynomials with rational integral coefficients of the type (1.3).

Then, we have the following result:

Corollary 2.4. *The inequality*

$$\sigma := \lim_{\substack{n_1 \to \infty \\ \cdots \\ n_m \to \infty}} \sigma_{n_1 \cdots n_m} \geqslant \lim_{\substack{n_1 \to \infty \\ \cdots \\ n_m \to \infty}} \prod_{i=1}^{m} \Delta_{n_i}^{-1/(2n_i(n_1 + \cdots + n_m))} \lim_{\substack{n_1 \to \infty \\ \cdots \\ n_m \to \infty}} \prod_{i=1}^{m} \left(\sum_{l_i=1}^{n_i} N_{i;\, l_i} \right)^{-1/(p(n_1 + \cdots + n_m))} \tag{2.7}$$

holds provided that the limits exist.

If $p=\infty$ and the weight functions are continuous, we write

$$\varrho_{n_1\cdots n_m}^{-(n_1+\cdots+n_m)} := \inf_{Q_{n_1\cdots n_m}} \max_{\substack{a_1\leqslant x_1\leqslant b_1\\ \cdots \\ a_m\leqslant x_m\leqslant b_m}} u_1(x_1)\cdots u_m(x_m)\,|Q_{n_1\cdots n_m}(x_1, ..., x_m)|,$$

where the infimum is extended over the same class of generalized polynomials as in (2.6).

According to Remark 2.2, we have the following:

COROLLARY 2.5. *The inequality*

$$\varrho := \lim_{\substack{n_1\to\infty\\ \cdots \\ n_m\to\infty}} \varrho_{n_1\cdots n_m}$$

$$\geqslant \lim_{\substack{n_1\to\infty\\ \cdots \\ n_m\to\infty}} \prod_{i=1}^{m} \Delta_{n_i}^{-1/(2n_i(n_1+\cdots+n_m))} \lim_{\substack{n_1\to\infty\\ \cdots \\ n_m\to\infty}} \prod_{i=1}^{m} \left(\sum_{l_i=1}^{n_i} \|\omega_{i;\,l_i}\|\right)^{-1/(n_1+\cdots+n_m)} \tag{2.8}$$

holds if the limits exist.

3. OPTIMALITY

We present two examples for which the inequalities (2.7) and (2.8) are optimal.

EXAMPLE 3.1. If $p=1$, $p_k(x_k)=(1-x_k^2)^{1/2}$, $u_k(x_k)=1$ and $[a_k, b_k]=[-1, 1]$, for all k, then the estimate (2.7) is optimal. We consider the system of functions $\{\phi_{k;\,j}(x_k)\}=\{\hat{U}_{j-1}(x_k)\}$, $j=1, ..., n_k$, $k=1, ..., m$, of normalized orthogonal Chebyshev polynomials of second kind with positive leading coefficient (as usual, we shall denote by $\tilde{R}_n(x)$ a polynomial of degree n normalized so that its leading coefficient is 1). Then

$$\sigma_{n_1\cdots n_m}^{-(n_1+\cdots+n_m)} = \inf_{\alpha_{k_1\cdots k_m}\in Z} \int_{-1}^{1}\cdots\int_{-1}^{1} \times\left|\sum_{k_1=1}^{n_1}\cdots\sum_{k_m=1}^{n_m} \alpha_{k_1\cdots k_m}\hat{U}_{k_1-1}(x_1)\cdots\hat{U}_{k_m-1}(x_m)\right| \times dx_1\cdots dx_m$$

$$\geqslant \|\tilde{U}_{n_1-1}\|_{2,p_1}^{-1} \cdots \|\tilde{U}_{n_m-1}\|_{2,p_m}^{-1}$$

$$\times \inf_{q \in \Pi} \int_{-1}^{1} \cdots \int_{-1}^{1} |\tilde{U}_{n_1-1}(x_1) \cdots \tilde{U}_{n_m-1}(x_m) + q|$$

$$\times dx_1 \cdots dx_m$$

$$= \|\tilde{U}_{n_1-1}\|_{2,p_1}^{-1} \cdots \|\tilde{U}_{n_m-1}\|_{2,p_m}^{-1}$$

$$\times \int_{-1}^{1} \cdots \int_{-1}^{1} |\tilde{U}_{n_1-1}(x_1) \cdots \tilde{U}_{n_m-1}(x_m)|\, dx_1 \cdots dx_m,$$

where $\|\cdot\|_{2,p_k}$ stands for the L_2-norm with weight $p_k(x_k)$, Π is the space of all algebraic polynomials with real coefficients in m real variables with total degree at most $n_1 + \cdots + n_m - m - 1$, and the last equality follows from [11]. Therefore

$$\sigma_{n_1 \cdots n_m}^{-(n_1 + \cdots + n_m)} \geqslant 2^m \left(\frac{2}{\pi}\right)^{m/2},$$

which implies

$$\sigma = \lim_{\substack{n_1 \to \infty \\ \cdots \\ n_m \to \infty}} \sigma_{n_1 \cdots n_m} \leqslant 1.$$

We have $\Delta_{n_k} = 1$ for all k, and

$$N_{i;j} = \int_{-1}^{1} |\hat{U}_j(x_i)|\, dx_i = 2(2/\pi)^{1/2},$$

for all i, j. Therefore the limit as $n_1 \to \infty, ..., n_m \to \infty$ of the right-hand side in (2.7) is also 1.

EXAMPLE 3.2. If $p_k(x_k) = (1 - x_k^2)^{-1/2}$, $u_k(x_k) = 1$, $[a_k, b_k] = [-1, 1]$ and $\{\phi_{k;j}(x_k)\} = \{\hat{T}_{j-1}(x_k)\}$, $j = 1, ..., n_k$, $k = 1, ..., m$, where $\hat{T}_{j-1}(x_k)$ is the normalized orthogonal Chebyshev polynomial with positive leading coefficient, then the estimate (2.8) is optimal. We actually have

$$\varrho_{n_1 \cdots n_m}^{-(n_1 + \cdots + n_m)}$$
$$= \inf_{\alpha_{k_1 \dots k_m} \in Z} \max_{\substack{-1 \leqslant x_1 \leqslant 1 \\ \cdots \\ -1 \leqslant x_m \leqslant 1}} \left| \sum_{k_1=1}^{n_1} \cdots \sum_{k_m=1}^{n_m} \alpha_{k_1 \dots k_m} \hat{T}_{k_1 - 1}(x_1) \cdots \hat{T}_{k_m - 1}(x_m) \right|$$
$$\geqslant \| \tilde{T}_{n_1 - 1} \|_{2, p_1}^{-1} \cdots \| \tilde{T}_{n_m - 1} \|_{2, p_m}^{-1} \inf_{r \in \Pi} \max_{\substack{-1 \leqslant x_1 \leqslant 1 \\ \cdots \\ -1 \leqslant x_m \leqslant 1}} |x_1^{n_1 - 1} \cdots x_m^{n_m - 1} + r|$$
$$= \| \tilde{T}_{n_1 - 1} \|_{2, p_1}^{-1} \cdots \| \tilde{T}_{n_m - 1} \|_{2, p_m}^{-1} \| \tilde{T}_{n_1 - 1} \cdots \tilde{T}_{n_m - 1} \|_{\infty, 1} \geqslant \left(\frac{1}{\pi} \right)^{m/2},$$

where the last equality follows by [8]. Therefore

$$\varrho_{n_1 \cdots n_m} \leqslant \left(\frac{1}{\pi} \right)^{-m/(2(n_1 + \cdots + n_m))}$$

and

$$\varrho = \lim_{\substack{n_1 \to \infty \\ \cdots \\ n_m \to \infty}} \varrho_{n_1 \cdots n_m} \leqslant 1.$$

On the other hand, it is clear that $\Delta_{n_k} = 1$, for all k, and

$$\| \omega_{i;\, j} \| = \| \hat{T}_j \|_\infty = \max_{x_i \in [-1, 1]} |\hat{T}_j(x_i)| = \begin{cases} \sqrt{1/\pi} & \text{if } \ j = 0, \\ \sqrt{2/\pi} & \text{if } \ j \geqslant 1. \end{cases}$$

Therefore the limit as $n_1 \to \infty$, ..., $n_m \to \infty$ of the right-hand side in (2.8) is also 1.

4. APPLICATIONS OF THEOREM 1.1

In this section, we apply Theorem 1.1 to ordinary polynomials. This case comes up when the system of functions $\{\phi_{k;\, i}(x_k)\}$ is the power sequence

$$\phi_{k;\, i}(x_k) = x_k^{i-1}, \qquad (i = 1, ...)$$

since, the corresponding orthonormal system obtained by the Schmidt procedure, is an orthonormal system of polynomials $\{\omega_{k;\, i}(x_k)\}$ on the interval $[a_k, b_k]$.

By specializing $p_k(x_k)=u_k(x_k)=[(x_k-a_k)(b_k-x_k)]^{-1/2}$, $k=1, ..., m$, we obtain the following result for multivariate polynomials which generalizes Theorem 2 in [15]. Theorems 4-8 in [15] can be generalized in a similar way.

THEOREM 4.1. *There exists a non trivial polynomial*

$$Q_{n_1 \cdots n_m}(x_1, ..., x_m)=\sum_{k_1=0}^{n_1} \cdots \sum_{k_m=0}^{n_m} \alpha_{k_1 \cdots k_m} x_1^{k_1} \cdots x_m^{k_m} \tag{4.1}$$

with rational integral coefficients, such that

$$\begin{aligned}\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} \prod_{i=1}^{m} [(x_i-a_i)(b_i-x_i)]^{-1/2} \, |Q_{n_1 \cdots n_m}(x_1, ..., x_m)|^p \, dx_1 \cdots dx_m \\ \leqslant \prod_{i=1}^{m} \left[\left(\pi + 2^{p/2} \frac{\Gamma(1/2)\,\Gamma((p+1)/2)}{\Gamma(p/2+1)} n_i \right) \right. \\ \left. \times 2^{pn_i/(2n_i+2)} (n_i+1)^{p-1} \left(\frac{b_i-a_i}{4} \right)^{pn_i/2} \right].\end{aligned} \tag{4.2}$$

Proof. Since $\phi_{k;\,i}(x_k)=x_k^{i-1}$, $i=1, ..., n_k+1$, to apply Theorem 1.1 we only need to replace n_i by n_i+1 in (1.4). Moreover, the polynomials $\{\omega_{k;\,i}(x_k)\}$ are the Chebyshev polynomials $\{\hat{T}_{k;\,i}(x_k)\}$. In this case

$$\Delta_{n_i+1}=\pi^{n_i+1} 2^{n_i} \left(\frac{b_i-a_i}{4} \right)^{n_i(n_i+1)} \tag{4.3}$$

and for all i, j,

$$N_{i;\,j}=\left(\frac{2}{\pi} \right)^{p/2} \frac{\Gamma(1/2)\,\Gamma((p+1)/2)}{\Gamma(p/2+1)}, \qquad N_{i;\,1}=\pi^{1-p/2}. \tag{4.4}$$

Therefore,

$$\begin{aligned}(n_i+1)^{p-1} \, \Delta_{n_i+1}^{p/(2n_i+2)} \sum_{l_i=1}^{n_i+1} N_{i;\,l_i} \\ = \left(\pi + 2^{p/2} \frac{\Gamma(1/2)\,\Gamma((p+1)/2)}{\Gamma(p/2+1)} n_i \right) 2^{pn_i/(2n_i+2)} (n_i+1)^{p-1} \left(\frac{b_i-a_i}{4} \right)^{pn_i/2}\end{aligned}$$

and the conclusion follows. ∎

On the other hand if we take $p_k(x_k)=[(x_k-a_k)(b_k-x_k)]^{-1/2}$ and $u_k(x_k)=1$, for all k, we can assert the following:

THEOREM 4.2. *There exists a non trivial polynomial of the type* (4.1) *such that*

$$\max_{\substack{a_1 \leqslant x_1 \leqslant b_1 \\ \cdots \\ a_m \leqslant x_m \leqslant b_m}} |Q_{n_1 \cdots n_m}(x_1, ..., x_m)|$$

$$\leqslant 2^{(1/2)[n_1/(n_1+1) + \cdots + n_m/(n_m+1)]} \prod_{i=1}^{m} (1+\sqrt{2}n_i) \left(\frac{b_i - a_i}{4}\right)^{n_i/2}.$$

Proof. From Remark 2.2 we have

$$\max_{\substack{a_1 \leqslant x_1 \leqslant b_1 \\ \cdots \\ a_m \leqslant x_m \leqslant b_m}} |Q_{n_1 \cdots n_m}(x_1, ..., x_m)| \leqslant \prod_{i=1}^{m} \left(\Delta_{n_i+1}^{1/(2n_i+2)} \sum_{l_i=1}^{n_i+1} \|\hat{T}_{i;\, l_i}\| \right),$$

where $\|\hat{T}_{i;\,1}\| = 1/\sqrt{\pi}$ and $\|\hat{T}_{i;\,j}\| = \sqrt{2/\pi}$, $i = 1, ..., m$, $j = 2, ...$. Using (4.3), the conclusion follows.

5. PROOF OF THEOREM 1.2

The proof follows along the same lines as in the proofs of Theorem 1.1 above and Theorem 1 in [4] with the obvious modifications. We shall only point out that the sum in (2.1) is now extended over all indices $k_i \geqslant 1$ such that $k_1 + \cdots + k_m \leqslant h$. Also, if $D_{h,m}$ denotes the determinant of the matrix of the system of linear forms involved, we have (see [4])

$$D_{h,m} = \prod_{s=m-2}^{h-2} \left(\prod_{k=1}^{m} |A_{k;\, h-s-1}| \right)^{\binom{s}{m-2}}$$

$$= \prod_{r=1}^{h-m+1} \left(\prod_{k=1}^{m} \Delta_{k;\, r}^{1/2} \right)^{\binom{h-r-1}{m-2}} = \prod_{j=1}^{h-m+1} (b_{1;\, jj} \cdots b_{m;\, jj})^{\binom{h-j}{m-1}}. \quad (5.1)$$

Remark 5.1. If $p = 2$ and $p_k(x_k) = u_k(x_k)$ for all k, we have

$$I_h \leqslant \left[\prod_{r=1}^{h-m+1} \left(\prod_{k=1}^{m} \Delta_{k;\, r} \right)^{\binom{h-r-1}{m-2}} \right]^{\binom{h}{m}^{-1}} \binom{h}{m} \quad (5.2)$$

(see [4]).

Remark 5.2. In the uniform case (see [4, 18]), we get

$$\max_{\substack{a_1 \leqslant x_1 \leqslant b_1 \\ \cdots \\ a_m \leqslant x_m \leqslant b_m}} u_1(x_1) \cdots u_m(x_m) \, |Q_h(x_1, ..., x_m)|$$

$$\leqslant \left[\prod_{r=1}^{h-m+1} \left(\prod_{k=1}^{m} \Delta_{k;r}^{1/2} \right)^{\binom{h-r-1}{m-2}} \right]^{\binom{h}{m}^{-1}} \sum_{\substack{l_1 + \cdots + l_m \leqslant h \\ l_i \geqslant 1}} \|\omega_{1;l_1}\| \cdots \|\omega_{m;l_m}\|, \tag{5.3}$$

where $\|\omega_{i;j}\| = \max_{a_i \leqslant x_i \leqslant b_i} u_i(x_i) \, |\omega_{i;j}(x_i)|$.

6. APPLICATIONS OF THEOREM 1.2

Let $\phi_{k;i}(x_k)$, $p_k(x_k)$ and $u_k(x_k)$ be the same as in the proof of Theorem 4.1. Then

$$b_{k;11} = \pi^{1/2}$$

and

$$b_{k;ii} = (2\pi)^{1/2} \left(\frac{b_k - a_k}{4} \right)^{i-1}, \qquad k = 1, ..., m, \quad i = 2, ... \tag{6.1}$$

Therefore

$$D_{h,m} = (2^{m/2})^{\binom{h-1}{m}} (\pi^{m/2})^{\binom{h}{m}} \left[\left(\frac{b_1 - a_1}{4} \right) \cdots \left(\frac{b_m - a_m}{4} \right) \right]^{\binom{h}{m+1}}. \tag{6.2}$$

Moreover for $k = 1, ..., m$, $i = 2, ...,$ the values of $N_{k;1}$ and $N_{k;i}$ are as in (4.4).

We observe that in this case, the expression (1.6) defines a polynomial of total degree $\leqslant h - m$. Therefore, applying Theorem 1.2, we have the following results to multivariate polynomials with m-unknowns and total degree $\leqslant n$:

THEOREM 6.1. *There exists a non trivial polynomial*

$$Q_n(x_1, ..., x_m) = \sum_{\substack{k_1 + \cdots + k_m \leqslant n \\ k_i \geqslant 0}} \alpha_{k_1 \cdots k_m} x_1^{k_1} \cdots x_m^{k_m}, \qquad m \geqslant 2, \tag{6.3}$$

with rational integral coefficients, such that

$$\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} \prod_{k=1}^{m} [(x_k - a_k)(b_k - x_k)]^{-1/2} |Q_n(x_1, ..., x_m)|^p \, dx_1 \cdots dx_m$$

$$\leqslant (2^{m/2})^{pn/(n+m)} (\pi^{m/2})^p \left[\left(\frac{b_1 - a_1}{4}\right) \cdots \left(\frac{b_m - a_m}{4}\right) \right]^{pn/(m+1)} \binom{n+m}{m}^{p-1}$$

$$\times \sum_{j=0}^{m} \binom{m}{j} \binom{n}{j} \pi^{(1-p/2)(m-j)} \left(\frac{2}{\pi}\right)^{pj/2} \left(\frac{\Gamma(1/2)\, \Gamma((p+1)/2)}{\Gamma(p/2+1)} \right)^j.$$

THEOREM 6.2. *There exists a non trivial polynomial of the type* (6.3) *such that*

$$\max_{\substack{a_1 \leqslant x_1 \leqslant b_1 \\ \cdots \\ a_m \leqslant x_m \leqslant b_m}} |Q_n(x_1, ..., x_m)|$$

$$\leqslant 2^{mn/(2(m+n))} \left[\left(\frac{b_1 - a_1}{4}\right) \cdots \left(\frac{b_m - a_m}{4}\right) \right]^{\frac{n}{m+1}} \sum_{j=0}^{m} \binom{m}{j} \binom{n}{j} 2^{j/2}.$$

Proof. By Remark 5.2

$$\max_{\substack{a_1 \leqslant x_1 \leqslant b_1 \\ \cdots \\ a_m \leqslant x_m \leqslant b_m}} |Q_h(x_1, ..., x_m)|$$

$$\leqslant \left[\prod_{r=1}^{h-m+1} \left(\prod_{k=1}^{m} \Delta_{k;r}^{1/2} \right)^{\binom{h-r-1}{m-2}} \right]^{\binom{h}{m}^{-1}} \sum_{\substack{l_1 + \cdots + l_m \leqslant h \\ l_i \geqslant 1}} \|\hat{T}_{1;l_1}\| \cdots \|\hat{T}_{m;l_m}\|.$$

From this and (6.2), the conclusion follows. ∎

THEOREM 6.3. *There exists a non trivial polynomial of the type* (6.3) *such that*

$$\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} |Q_n(x_1, ..., x_m)| \, dx_1 \cdots dx_m$$

$$\leqslant 2^m \left(\frac{b_1 - a_1}{2}\right) \cdots \left(\frac{b_m - a_m}{2}\right)$$

$$\times \left[\left(\frac{b_1 - a_1}{4}\right) \cdots \left(\frac{b_m - a_m}{4}\right) \right]^{n/(m+1)} \binom{n+m}{m}.$$

Proof. If we consider $p_k(x_k)=[(x_k-a_k)(b_k-x_k)]^{1/2}$ and $u_k(x_k)=1$, for all k, then

$$b_{k;\,ii}=(\pi/2)^{1/2}\left(\frac{b_k-a_k}{2}\right)\left(\frac{b_k-a_k}{4}\right)^{i-1}, \qquad k=1, ..., m, \quad i=1, ... \tag{6.4}$$

and, for all k, i,

$$N_{k;\,i}=2(2/\pi)^{1/2}.$$

Now, applying Theorem 1.2, the theorem follows. ∎

THEOREM 6.4. *There exists a non trivial polynomial of the type* (6.3) *such that*

$$\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\prod_{k=1}^{m}[(x_k-a_k)(b_k-x_k)]^{1/2}\,Q_n^2(x_1, ..., x_m)\,dx_1\cdots dx_m$$
$$\leqslant\left(\frac{\pi}{2}\right)^m\left(\frac{b_1-a_1}{2}\right)^2\cdots\left(\frac{b_m-a_m}{2}\right)^2$$
$$\times\left[\left(\frac{b_1-a_1}{4}\right)\cdots\left(\frac{b_m-a_m}{4}\right)\right]^{2n/(m+1)}\binom{n+m}{m}.$$

Proof. In this case, we consider $p_k(x_k)=u_k(x_k)=[(x_k-a_k)(b_k-x_k)]^{1/2}$, $k=1, ..., m$. Taking into account (6.4), the theorem follows from Remark 5.1. ∎

THEOREM 6.5. *There exists a non trivial polynomial of the type* (6.3) *such that*

$$\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\prod_{k=1}^{m}\left(\frac{b_k-x_k}{x_k-a_k}\right)^{1/2}Q_n^2(x_1, ..., x_m)\,dx_1\cdots dx_m$$
$$\leqslant\pi^m\left(\frac{b_1-a_1}{2}\right)\cdots\left(\frac{b_m-a_m}{2}\right)$$
$$\times\left[\left(\frac{b_1-a_1}{4}\right)\cdots\left(\frac{b_m-a_m}{4}\right)\right]^{2n/(m+1)}\binom{n+m}{m}.$$

Proof. Let $p_k(x_k)=u_k(x_k)=((b_k-x_k)/(x_k-a_k))^{1/2}$, $k=1, ..., m$. Then

$$b_{k;\,ii}=\pi^{1/2}\left(\frac{b_k-a_k}{2}\right)^{1/2}\left(\frac{b_k-a_k}{4}\right)^{i-1}, \qquad k=1, ..., m, \quad i=1, ...$$

Therefore, from (5.2) the conclusion follows. ∎

Remark 6.6. If we consider the systems $\{\phi_{k;\,i}(x_k)=x_k^{i-1}\}_{i=1}^{n+1}$ on the segments $[a_k, b_k]=[a, b]$ and the weight functions $p_k(x_k)=p(x_k)$, for all k, then, for a g fixed the polynomials $\omega_{k;\,g}(x_k)=\omega_g(x_k)$ and the Gram determinants $\Delta_{k;\,g}=\Delta_g$ are the same for all k. To compute the determinant $D_{n,m}$ of the matrix which transforms the monomials $\{x_1^{e_1}\cdots x_m^{e_m}\}$ of total degree $\leqslant n$, into the products $\{\omega_{1;\,l_1}(x_1)\cdots\omega_{m;\,l_m}(x_m)\}$ of total degree $\leqslant n$, we observe that $k_i=e_i+1$ and, therefore, the condition $e_1+\cdots+e_m\leqslant n$ becomes $k_1-1+\cdots+k_m-1\leqslant n$ which implies $h=n+m$. Moreover, $b_{k;\,ii}=b_{ii}$, for all k. From this and (5.1), we have

$$D_{n,m}=\prod_{r=1}^{n+1} b_{rr}^{m\binom{n+m-r}{m-1}}.$$

For example, if $p(x_k)=u(x_k)=[(x_k-a)(b-x_k)]^{-1/2}$, $k=1, ..., m$, and $p=2$, Remark 5.1 together with (4.3) yields the estimate

$$\int_a^b\cdots\int_a^b \prod_{k=1}^{m} [(x_k-a)(b-x_k)]^{-1/2}\, Q_n^2(x_1, ..., x_m)\, dx_1\cdots dx_m$$
$$\leqslant \pi^m 2^{nm/(n+m)}\left(\frac{b-a}{4}\right)^{2nm/(m+1)}\binom{n+m}{m}. \tag{6.5}$$

If $b_k-a_k<4$, for all k, Theorems 4.1, 4.2, 6.1–6.5, and the inequality (6.5) show the existence of multivariate polynomials with rational integral coefficients, not simultaneously zero, with arbitrarily small norms. Moreover, Theorems 6.1–6.5 show that such polynomials exist if $(b_1-a_1)\cdots(b_m-a_m)<4^m$.

On the other hand, Theorems 6.1, 6.3, 6.4, and 6.5 are extensions to several variables of Theorems 2, 4, 5, and 6 in [15], respectively. Theorems 4.2 and 6.2 above extends Fekete's Theorem (see [9,4]).

7. PROOF OF THEOREM 1.3

By Remark 2.1, we have

$$I_{n\cdots n}=\int_0^1\cdots\int_0^1 Q_{n\cdots n}^2(x_1, ..., x_m)\, dx_1\cdots dx_m\leqslant (n+1)^m \prod_{k=1}^{m} \Delta_{k;\,n+1}^{1/(n+1)}.$$

From the last inequality and the definition of $v_{n\cdots n}^{-2mn}(\gamma_1, ..., \gamma_m)$, we have

$$v_{n\cdots n}(\gamma_1, ..., \gamma_m)\geqslant (n+1)^{-1/(2n)}\left(\prod_{k=1}^{m} \Delta_{k;\,n+1}\right)^{-1/(2mn(n+1))}. \tag{7.1}$$

Since $\Delta_{k;\,n+1}$ is the Gram determinant of the system of functions $\{x_k^{i^{\gamma_k}}\}_{i=0}^n$ we have

$$\left(\prod_{k=1}^{m} \Delta_{k;\,n+1}\right)^{-1/(2mn(n+1))}$$
$$= \prod_{k=1}^{m} \left[\prod_{i=0}^{n} (1+2i^{\gamma_k}) \prod_{\substack{0\\ i\neq s}}^{n} \left|\frac{1+i^{\gamma_k}+s^{\gamma_k}}{i^{\gamma_k}-s^{\gamma_k}}\right|\right]^{1/(2mn(n+1))}$$
$$\geqslant \exp\left\{\sum_{k=1}^{m}\left[\frac{1}{2mn(n+1)} \sum_{\substack{0\\ i\neq s}}^{n} \ln\left|\frac{(i/n)^{\gamma_k}+(s/n)^{\gamma_k}}{(i/n)^{\gamma_k}-(s/n)^{\gamma_k}}\right|\right]\right\}. \tag{7.2}$$

Passing to the limit in (7.1) when $n\to\infty$, and taking into account (7.2), we obtain

$$\mathbf{v}(\gamma_1, ..., \gamma_m) \geqslant \exp\left[\sum_{k=1}^{m}\left(\frac{1}{2m}\int_0^1\int_0^1 \ln\left|\frac{x^{\gamma_k}+y^{\gamma_k}}{x^{\gamma_k}-y^{\gamma_k}}\right| dx\, dy\right)\right]. \tag{7.3}$$

It is known that

$$\frac{1}{2}\int_0^1\int_0^1 \ln\left|\frac{x^{\gamma_k}+y^{\gamma_k}}{x^{\gamma_k}-y^{\gamma_k}}\right| dx\, dy = \sum_{r=0}^{\infty} \frac{1}{(2r+1)[(2r+1)\gamma_k+1]} = J(\gamma_k) \tag{7.4}$$

(see [2]). The conclusion follows from (7.3) and (7.4). ∎

Remark 7.1. Since $J(\gamma_k)>0$, for all k, we have

$$\mathbf{v}(\gamma_1, ..., \gamma_m) > 1$$

and then

$$\lim_{n\to\infty} v_{n\cdots n}^{-2mn}(\gamma_1, ..., \gamma_m) = 0.$$

Consequently for each m-tuple of positive real numbers $\gamma_1, ..., \gamma_m$, there exist quasi-polynomials $Q_{n\cdots n}(x_1, ..., x_m)$ of the type (1.8) with arbitrarily small quadratic norms.

8. EXAMPLES

We consider some particular cases.

(a) If $\gamma_k = 2$, for all k, then $J(2) = \sum_{r=0}^{\infty} 2/((4r+2)(4r+3)) = \pi/4 - \frac{1}{2}\ln 2$. Therefore,

$$\frac{e^{\pi/4}}{\sqrt{2}} \leqslant \mathbf{v}(2, ..., 2).$$

(b) If $\gamma_k = 1$, for all k, then $J(1) = \sum_{r=0}^{\infty} 1/((2r+1)(2r+2)) = \ln 2$, and from Theorem 1.3 we conclude that

$$2 \leqslant \mathbf{v}(1, ..., 1).$$

However, this bound can be improved. According to definition (1.9), we write

$$\rho := \rho[0, 1], \qquad \rho_n^{-n} := \rho_n^{-n}[0, 1]. \tag{8.1}$$

The existence of the limit ρ was proved by Schnirelman (see [7, 10, 16]). It is known that (see [7])

$$2.36 \leqslant \rho \leqslant 2.376... \,.$$

Therefore,

$$\begin{aligned} v_{n \cdots n}^{-2mn}(1, ..., 1) &\leqslant \int_0^1 \cdots \int_0^1 Q_n^2(x_1) \cdots Q_n^2(x_m)\, dx_1 \cdots dx_m \\ &\leqslant [\max_{0 \leqslant x_i \leqslant 1} Q_n^2(x_i)]^m = \rho_n^{-2mn}, \end{aligned}$$

where $0 \not\equiv Q_n(x) \in H_n^e$ is a polynomial such that its uniform deviation from zero on the interval $[0, 1]$ is least. We therefore have

$$2.36 \leqslant \rho \leqslant \mathbf{v}(1, ..., 1).$$

On the other hand, we can use the method by A. O. Gelfond and L. G. Schnirelman [13] to find an upper bound. For each positive integer n, let Ω_n denote the least common multiple of $\{1, 2, ..., n\}$, and let $Q_{n \cdots n}(x_1, ..., x_m)$ be a rational integral polynomial which deviate the least from zero (quadratic norm) among all rational integral polynomials of the type (1.8) with $n_i = n$ for all i. Then

$$\Omega_{2n+1}^m \int_0^1 \cdots \int_0^1 Q_{n \cdots n}^2(x_1, ..., x_m)\, dx_1 \cdots dx_m \geqslant 1$$

and

$$v_{n \cdots n}^{-2mn}(1, ..., 1) \geqslant \frac{1}{\Omega_{2n+1}^m}.$$

Hence

$$\ln v_{n\cdots n}(1, ..., 1) \leqslant \frac{\ln \Omega_{2n+1}}{2n}.$$

Since $\ln \Omega_n \leqslant \pi(n) \ln n$, for all n, where $\pi(n)$ denote the number of prime numbers less than or equal to n, we have

$$\ln \mathbf{v}(1, ..., 1) = \lim_{n\to\infty} \ln v_{n\cdots n}(1, ..., 1) \leqslant \lim_{n\to\infty} \frac{\pi(2n+1)\ln(2n+1)}{2n} = 1.$$

We conclude that

$$\mathbf{v}(1, ..., 1) \leqslant e.$$

(c) If $\gamma_k = 1/2$, for all k, then $J(1/2) = \sum_{r=0}^{\infty} 2/((2r+1)(2r+3)) = 1$. Therefore

$$\mathbf{v}(\tfrac{1}{2}, ..., \tfrac{1}{2}) \geqslant e.$$

(d) If $\gamma_1 = 1$ and $\gamma_2 = 1/2$, then

$$\mathbf{v}(1, \tfrac{1}{2}) = \mathbf{v}(\tfrac{1}{2}, 1) \geqslant (2e)^{1/2}.$$

9. PROOF OF THEOREM 1.4

The proof being essentially the same for any segment $[a, b]$, we shall restrict to the interval $[0, 1]$. Let $R_n(x)$ and $Q_n(x)$ be two polynomials belonging to H_n^e of minimal diophantine uniform and L_2 deviation from zero in $[0, 1]$, respectively, that is, such that

$$\rho_n^{-n} = \|R_n\|_\infty \qquad \text{and} \qquad \tau_n^{-2n} := \tau_n^{-2n}[0, 1] = \|Q_n\|_2^2.$$

Then

$$\|Q_n\|_2^2 \leqslant \|R_n\|_2^2 \leqslant \|R_n\|_\infty^2 \int_0^1 p(x)\, dx,$$

and, therefore,

$$\tau_n^{-2n} \leqslant M\rho_n^{-2n}, \tag{9.1}$$

where $M = \int_0^1 p(x)\, dx$.

To obtain a lower bound for τ_n^{-2n}, we consider the function $Q_n^2(x)$. The Fourier series for this function is

$$Q_n^2(x) = \sum_{k=0}^{2n} c_k \omega_k(x) \qquad \left(c_k = \int_0^1 Q_n^2(x)\, \omega_k(x)\, p(x)\, dx\right).$$

The function $Q_n^2(x)$ is continuous on the segment $[0, 1]$. It therefore achieves its maximum value in this segment, so that there exists a point $t \in [0, 1]$ such that

$$\rho_n^{-2n} \leqslant \|Q_n\|_\infty^2 = Q_n^2(t).$$

But at the point t we have

$$Q_n^2(t) = \int_0^1 K_{2n}(t, x)\, Q_n^2(x)\, p(x)\, dx,$$

and, therefore,

$$\rho_n^{-2n} \leqslant K_{2n} \int_0^1 Q_n^2(x)\, p(x)\, dx = K_{2n} \tau_n^{-2n}, \tag{9.2}$$

where $K_{2n} := K_{2n}[0, 1]$. Thus, by (9.1) and (9.2), we have

$$M^{-1/(2n)} \rho_n \leqslant \tau_n \leqslant K_{2n}^{1/(2n)} \rho_n. \tag{9.3}$$

Since $p(x) \in \mathscr{P}[0, 1]$, the inequalities (9.3) and the existence of ρ in (8.1) show the equalities (1.10) and (1.11) for the segment $[0, 1]$. ∎

REFERENCES

1. E. Aparicio Bernardo, On some properties of polynomials with integer coefficients, and on approximation of functions in the mean by polynomials with integer coefficients, *Izv. Akad. Nauk SSSR Ser. Mat.* **19** (1955), 303–318. [In Russian]
2. E. Aparicio Bernardo, On least deviation from zero of quasi-polynomials with integral algebraic coefficients, *Bull. Moscow State Univ. Ser. I Math. Mech.* **2** (1962), 21–32. [In Russian]
3. E. Aparicio Bernardo, Método de las formas lineales para la acotación de las desviaciones mínimas a cero de los polinomios generalizados de coeficientes enteros, *in* "Proc. 1st Conference of Portuguese and Spanish Mathematicians, Lisbon, 1972," pp. 133–143, Inst. Jorge Juan Mat., Madrid, 1973.
4. E. Aparicio Bernardo, Generalization of a theorem of M. Fekete to polynomials with integer coefficients in several unknowns, *Rev. Mat. Hisp. Amer. (4)* **36** (1976), 105–124. [In Spanish]
5. E. Aparicio Bernardo, On some results in the problem of diophantine approximation of functions by polynomials, *Proc. Steklov Inst. Math.* **4** (1985), 7–10.

6. E. Aparicio Bernardo, Generalization of the theorem of L. G. Schnirelman, *J. Analyse Math.* **48** (1987), 217–224.
7. P. Borwein and T. Erdelyi, The integer Chebyshev problem, *Math. Comp.* **65**, No. 214 (1996), 661–681.
8. H. Ehlich and K. Zeller, Cebysev-Polynome in mehreren Veranderlichen, *Mat. Zametki* **93** (1966), 142–143.
9. M. Fekete, Uber die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten, *Math. Z.* **17** (1923), 228–249.
10. Le Baron O. Ferguson, "Approximation by Polynomials with Integral Coefficients," Math. Surveys, Vol. 17, Amer. Math. Soc., Providence, RI, 1980.
11. J. Fromm, L_1-approximation to zero, *Math. Z.* **151** (1976), 31–33.
12. I. M. Gelfand, "Lectures on Linear Algebra," Dover, New York, 1961.
13. A. O. Gelfond, Commentary on the papers on the estimation of the number of primes not exceeding a given value and on prime numbers, *in* "Collected Works of P. L. Chebyshev," Vol. 1, pp. 285–288, Akad. Nauk SSSR, Moscow/Leningrad, 1946. [In Russian]
14. D. Hilbert, Ein Beitrag zur Theorie des Legendre'schen Polynoms, *Acta Math.* **18** (1894), 155–159.
15. F. Luquin, L_p-deviations from zero of polynomials with integral coefficients, *Acta Arith.* **69**, No. 3 (1995), 217–228.
16. H. L. Montgomery, "Ten Lectures on the Interface between Analytic Number Theory and Harmonic Analysis," CBMS, Regional Conference Series in Mathematics, Vol. 84, Amer. Math. Soc., Providence, RI, 1994.
17. I. P. Natanson, "Constructive Function Theory," Vols. I, II, Ungar, New York, 1964–1965.
18. I. N. Sanov, Functions with integral parameters deviating the least from zero, *Leningrad. Gos. Univ. Uhcen. Zap. Ser. Mat. Nauk* **111** (1949), 32–46. [In Russian]
19. W. M. Schmidt, "Diophantine Approximation," Lecture Notes in Math., Vol. 785, Springer-Verlag, New York/Berlin, 1980.
20. G. A. Zirnova, On the approximation of functions by polynomials with integral coefficients, *Izv. Vyssh. Uchebn. Zaved. Mat.* **4**, No. 5 (1958), 80–88. [In Russian]